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LINE INTEGRALS IN APPLIED WELFARE ECONOMICS: A SUMMARY OF BASIC THEOREMS

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Discussion Paper No. 54-95

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Abstract

The main objective of this paper is a systematic representation of the basic theorems on line integrals to derive consistent and unique measures of welfare change. We describe the conditions of path independence, starting from the theoretical background of the Marshallian demand function, and give the mathematical foundation for Schwarz's Theorem by referring to the construct of a gradient field. Thus we substantially proceed beyond Takayama's pertinent contributions which, centering on the implications of Hicksian demand theory, pay little attention to the discussion of the problem of path independence. Moreover, his considerations on path independence are incomplete so that essential relationships of formal aspects cannot be perceived.

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1 The Problem

Applied welfare economics derives measures of the change in the well-being for an individual household between two situations which refer to different bundles of commodities consumed. From these measures, by aggregation, we deduce measures of welfare change for an economy.

Following Boadway and Bruce (1984, pp. 195–201), the problem of measuring the welfare change of a household may be presented as follows. The household's decision problem is to choose commodity quantities $x = (x_1, \dots, x_n)$ in order to maximize its ordinal, strictly quasi-concave utility function $U = u(x)$ given a budget constraint ($p = (p_1, \dots, p_n) > 0$ is the vector of given commodity prices, y is the given household income):

$$\max_x u(x_1, \dots, x_n) \qquad \text{subject to} \qquad \sum_{j=1}^{n} p_j x_j = y. \tag{1}$$

The solution to this problem gives a set of Marshallian (uncompensated) demand functions $x_j^M(p,y)$ and a value for the Lagrangean multiplier $\lambda(p,y) > 0$ representing the marginal utility of income.

The pertinent indirect utility function is

$$v(p,y) = u\left[x_1^M(p,y), \dots, x_n^M(p,y)\right], \tag{2}$$

which satisfies Roy's Theorem:

$$x_j^M(p,y) = -\frac{\partial v(p,y)/\partial p_j}{\partial v(p,y)/\partial y} \qquad (1 \le j \le n). \tag{3}$$

Recalling the envelope theorem, we may note

$$\lambda(p,y) = \frac{\partial v(p,y)}{\partial y}. \tag{4}$$

The change in (indirect) utility can be derived by total differentiation and the application of Roy's Theorem and the envelope theorem as

$$\begin{aligned} dv &= \sum_{j=1}^{n} \frac{\partial v}{\partial p_j} dp_j + \frac{\partial v}{\partial y} dy \\ &= -\lambda \sum_{j=1}^{n} x_j^M(p,y) dp_j + \lambda dy. \end{aligned} \tag{5}$$

If we divide through by λ we obtain a monetary measure of welfare change dW for differential changes in the exogenous prices and income faced by the consumer.

$$dW = \frac{dv}{\lambda} = -\sum_{j=1}^{n} x_j^M(p,y) dp_j + dy. \tag{6}$$

Now suppose prices and income change by discrete amounts between two situations (1) and (2). In this instance we must integrate the differential welfare changes between the initial situation (1) and the final situation (2) to get

$$\Delta W = \int_{(1)}^{(2)} dW = -\sum_{j=1}^{n} \int_{(1)}^{(2)} x_j^M(p, y)\, dp_j + \Delta y. \tag{7}$$

For the case of two commodities consumed we obtain

$$\Delta W = -\left[\int_{(1)}^{(2)} x_1^M(p_1, p_2, y)\, dp_1 + \int_{(1)}^{(2)} x_2^M(p_1, p_2, y)\, dp_2\right] + \Delta y.$$

The right-hand side of (7) is called a line integral or contour integral (cf. Hotelling (1938), Silberberg (1972), Apostol (1974)) which is the sum of several integrals each one of which depends on the values of variables (here prices) which are variables of integration in the other integrals.

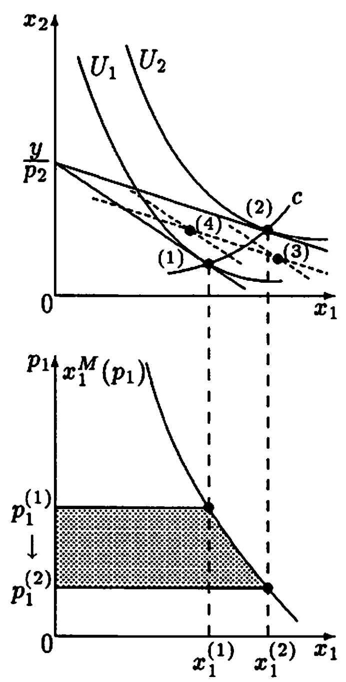

Fig. 1

Before discussing the fundamental problem of aggregation over commodities inherent in relationship (7), we shall look at two implications of (7). On the one hand, if the household's income y changes by Δy with all prices constant, ΔW is equal to Δy.

On the other hand, if the price p_1 changes while all other prices and income are held constant, the welfare change turns out to be

$$\Delta W = -\int_{p_1^{(1)}}^{p_1^{(2)}} x_1^M(p, y)\, dp_1. \tag{8}$$

This measure is illustrated in Fig. 1 for the case of two commodities x_1 and x_2 (the latter may be substituted by a composite of all goods other than x_1 in the case of several commodities in the sense of Hicks (1946)). In the upper diagram the price of x_1 is assumed to fall from $p_1^{(1)}$ to $p_1^{(2)}$. The integral in (8) is evaluated by moving from equilibrium (1) to equilibrium (2) along the price consumption line denoted c to which the Marshallian demand curve $x_1^M(p_1)$ corresponds in the lower diagram. Here the welfare change measure ΔW of (8) usually referred to as the Marshallian consumer surplus is given by the shaded area. (To be precise, this welfare measure was developed by Dupuit (1844) at first.)

The generalization of the measure of consumer surplus to multi-price changes (and the change of income) means the evaluation of (7) by allowing each of the prices (and income) to change from situation (1) to (2). Then the value of the integral for any one price change and of the sum of the integrals in (7) will, in general, depend on the path along which the integration is done. The integral is called path dependent, implying that the measure of welfare change ΔW is not well-defined (for intuitive arguments cf. also Hicks (1956), Glaister (1981), Ng (1983) and Auerbach (1985)).

"Therefore, there is a fundamental ambiguity involved in obtaining monetary measures of welfare change which can only be resolved by arbitrarily selecting a particular path of prices and income in going from one situation to another." (Boadway and Bruce (1984), p. 199). In addition, these authors correctly stress (pp. 199–200) that the uniqueness of the Marshallian consumer surplus according to (8), in a very strict sense, may also be taken as artificial, since there exist several ways of getting from point (1) to (2) (cf. Fig. 1) by combinations of price and income changes; each of them gives a different measure of welfare change according to the path selected (e.g., path (1) $\rightarrow$ (3) $\rightarrow$ (2) or path (1) $\rightarrow$ (4) $\rightarrow$ (2)).

The main objective of our paper is a systematic representation of the formal instruments related to line integrals to derive consistent an unique measures of welfare change. We describe the conditions of path independence, starting from the theoretical background of the Marshallian demand function, and give the mathematical foundation for Schwarz's Theorem by referring to the construct of a gradient field. Therefore we substantially proceed beyond Takayama's contributions (1984,1987) which, centering on the implications of Hicksian demand theory, pay little attention to the discussion of the problem of path independence. Moreover, his considerations on path independence are incomplete so that essential relationships of formal aspects cannot be perceived.

Thus we shall present basic theorems on the line integral, omitting the proofs (for a detailed formal analysis cf. Heuser (1993, pp. 349–407)), with the aim of deriving the mathematical foundation for uniqueness of the value of the line integral (integrability conditions). Our considerations will be enriched by references to demand theory in the form of examples, paying special attention to the condition of path independence (cf. also Glaister (1981), pp. 25–29).

2 Rectifiable Paths

At first we shall deal with the interval to which the solution of a line integral refers. The continuous image of a compact interval is a curve which is a compact and connected subset of $\mathbb{R}^n$ ($n \in \mathbb{N}$). Such a curve is described by a so-called path which we shall define as follows.

2.1 Definition. *Let* $[a, b] \subset \mathbb{R}$ *be an interval. Then a continuous map* $\gamma : [a, b] \rightarrow \mathbb{R}^n$ *is called a* path *in* $\mathbb{R}^n$. *The set* $\Gamma_\gamma := \{\gamma(h) \mid h \in [a, b]\} \subset \mathbb{R}^n$ *is called the* arc *belonging to path* γ. *The points* $\gamma(a)$ *and* $\gamma(b)$ *are designated as* initial point *and* endpoint *of* γ, *respectively.*

Here are two examples for different paths.

2.2 Examples.

(a) A household purchases $n \in \mathbb{N}$ commodities with prices p_j $(1 \leq j \leq n)$. If the interval $[a, b] \subset \mathbb{R}$ refers to two different price structures and if $(p_1^{(1)}, \dots, p_n^{(1)})$ and $(p_1^{(2)}, \dots, p_n^{(2)})$ are the price vectors at the situations considered, respectively, then the way the price vector has to go from $(p_1^{(1)}, \dots, p_n^{(1)})$ to $(p_1^{(2)}, \dots, p_n^{(2)})$ is a path $\gamma : [a, b] \rightarrow \mathbb{R}^n$ in the sense of the definition given above.

(b) Be $\gamma : [a,b] \to \mathbb{R}^n$ a path. Then

$$\gamma^- : [a,b] \to \mathbb{R}^n, \qquad [a,b] \ni h \mapsto \gamma(a+b-h)$$

is a path with $\Gamma_{\gamma^-} = \Gamma_\gamma$. In addition, the initial point of γ^- is the endpoint of γ and the endpoint of γ^- is the initial point of γ. Therefore we call γ^- the *path inverse to* γ.

In most practical cases we find a particular type of paths: the rectifiable paths. Grossly spoken, they are characterized by a finite length.

In the following considerations we use $\|\cdot\|$ for the Euclidean norm in $\mathbb{R}^n$, i.e. $\|(w_1,\dots,w_n)\| = \sqrt{\sum_{i=1}^n w_i^2}$ $((w_1,\dots,w_n) \in \mathbb{R}^n)$. Moreover, let $\mathcal{Z}$ be the set of all finite partitions of the interval $[a,b]$; thus

$$\mathcal{Z} = \{(h_0,h_1,\dots,h_m) \mid m \in \mathbb{N}, a = h_0 < h_1 < \dots < h_m = b\}.$$

2.3 Definition. *A path* $\gamma : [a,b] \to \mathbb{R}^n$ *is called* rectifiable, *if there exists a constant* $M \in \mathbb{R}$, *such that for any partition* $Z := (h_0,h_1,\dots,h_m) \in \mathcal{Z}$ *always*

$$\mathcal{L}(\gamma, Z) := \sum_{k=1}^{m} \|\gamma(h_k) - \gamma(h_{k-1})\| \leq M.$$

In this case

$$\mathcal{L}(\gamma) := \sup_{Z \in \mathcal{Z}} \mathcal{L}(\gamma, Z) \in \mathbb{R}_+$$

is called the path length *of* γ.

We should mention that a path γ is rectifiable if, and only if, the corresponding inverse path γ^- is rectifiable.

If $\gamma^{(1)} : [a,b] \to \mathbb{R}^n$, $\gamma^{(2)} : [c,d] \to \mathbb{R}^n$ are two paths with $\gamma^{(1)}(b) = \gamma^{(2)}(c)$ (i.e. the endpoint of $\gamma^{(1)}$ coincides with the initial point of $\gamma^{(2)}$), then also $\gamma : [a, b+(d-c)] \to \mathbb{R}^n$ is a path such that

$$\gamma(h) := \begin{cases} \gamma^{(1)}(h) & \text{if } h \in [a,b], \\ \gamma^{(2)}(h+c-b) & \text{if } h \in [b, b+(d-c)]. \end{cases}$$

We call γ the *path composed of the sections* $\gamma^{(1)}$ *and* $\gamma^{(2)}$: $\Gamma_\gamma = \Gamma_{\gamma^{(1)}} \cup \Gamma_{\gamma^{(2)}}$. Usually, the composite path is designated as $\gamma^{(1)}\gamma^{(2)}$. By this procedure we may inductively combine $m \in \mathbb{N}$ sectional paths for which corresponding initial points and endpoints are identical.

2.4 Theorem. *Suppose that* $\gamma^{(1)} : [a,b] \to \mathbb{R}^n$, $\gamma^{(2)} : [c,d] \to \mathbb{R}^n$ *are two paths subject to* $\gamma^{(1)}(b) = \gamma^{(2)}(c)$. *Then the composite path* $\gamma^{(1)}\gamma^{(2)}$ *will be rectifiable if, and only if,* $\gamma^{(1)}$ *and* $\gamma^{(2)}$ *are rectifiable. In this case the length of* $\gamma^{(1)}\gamma^{(2)}$ *turns out to be*

$$\mathcal{L}(\gamma^{(1)}\gamma^{(2)}) = \mathcal{L}(\gamma^{(1)}) + \mathcal{L}(\gamma^{(2)}).$$

Now we consider the case that the path $\gamma : [a,b] \to \mathbb{R}^n$ is continuously differentiable. The following theorem indicates that in this case the length of γ may be explicitly calculated. The symbol γ' denotes the derivative of γ with respect to h.

2.5 Theorem. *If $\gamma := (\gamma_1, \ldots, \gamma_n) : [a, b] \to \mathbb{R}^n$ is a continuously differentiable path, then γ is rectifiable and we obtain*

$$\mathcal{L}(\gamma) = \int_a^b \|\gamma'(h)\| \, dh = \int_a^b \sqrt{\sum\nolimits_{j=1}^n (\gamma_j'(h))^2} \, dh.$$

Subsequently, we shall nearly exclusively encounter continuously differentiable or at least piecewise continuously differentiable paths. A path γ is called *piecewise continuously differentiable*, if there exists a finite number of continuously differentiable paths $\gamma_1, \ldots, \gamma_m$ so that we get $\gamma = \gamma_1\gamma_2\cdots\gamma_m$. A continuously differentiable path is also termed as *smooth*. Correspondingly, we refer to a piecewise continuously differentiable path as a *piecewise smooth path*. From Theorems 2.4 and 2.5 immediately results

2.6 Theorem. *Each piecewise smooth path $\gamma = \gamma^{(1)} \cdots \gamma^{(m)}$ is rectifiable so that*

$$\mathcal{L}(\gamma) = \sum_{k=1}^m \mathcal{L}(\gamma^{(k)}).$$

2.7 Example. We shall now return to example 2.2 (a) and construct a special path from $(p_1^{(1)}, \ldots, p_n^{(1)})$ to $(p_1^{(2)}, \ldots, p_n^{(2)})$ by indicating specific sectional paths. For this purpose we define $\gamma^{(j)} : [p_j^{(1)}, p_j^{(2)}] \to \mathbb{R}^n$ $(1 \le j \le n)$ by

$$\gamma^{(j)}(h) := (p_1^{(2)}, \ldots, p_{j-1}^{(2)}, h, p_{j+1}^{(1)}, \ldots, p_n^{(1)}) \qquad (h \in [p_j^{(1)}, p_j^{(2)}]).$$

Obviously, the sections $\gamma^{(j)}$ $(1 \le j \le n)$ are smooth paths (here we are confronted with straight lines!); the endpoint of $\gamma^{(j)}$ coincides with the initial point of $\gamma^{(j+1)}$ $(1 \le j \le n-1)$. Thus the composite path $\gamma^{(1)} \cdots \gamma^{(n)}$ is a piecewise smooth path from $(p_1^{(1)}, \ldots, p_n^{(1)})$ to $(p_1^{(2)}, \ldots, p_n^{(2)})$ ($\gamma^{(1)} \cdots \gamma^{(n)}$ is a finite polygonal curve from $(p_1^{(1)}, \ldots, p_n^{(1)})$ to $(p_1^{(2)}, \ldots, p_n^{(2)})$).

On the basis of Theorems 2.5 and 2.6 the length of $\gamma^{(1)} \cdots \gamma^{(n)}$ amounts to

$$\mathcal{L}(\gamma^{(1)} \cdots \gamma^{(n)}) = \sum_{j=1}^n \mathcal{L}(\gamma^{(j)}) = \sum_{j=1}^n \int_{p_j^{(1)}}^{p_j^{(2)}} \|(\gamma^{(j)})'(h)\| \, dh.$$

From the definition of $\gamma^{(j)}$ immediately follows

$$(\gamma^{(j)})'(h) = (0, \ldots, 0, 1, 0, \ldots, 0) \qquad (h \in [p_j^{(1)}, p_j^{(2)}\ ,])$$

where the number 1 is located at the j-th place $(1 \le j \le n)$. Thus we obtain in all (as expected according to the definition of $\gamma^{(1)} \cdots \gamma^{(n)}$):

$$\mathcal{L}(\gamma^{(1)} \cdots \gamma^{(n)}) = \sum_{j=1}^n \int_{p_j^{(1)}}^{p_j^{(2)}} 1 \, dh = \sum_{j=1}^n (p_j^{(2)} - p_j^{(1)}).$$

3 Line Integrals

We shall define line integrals as follows.

3.1 Definition. *Let* $\gamma := (\gamma_1, \dots, \gamma_n) : [a, b] \to \mathbb{R}^n$ *be a rectifiable path and* $f := (f_1, \dots, f_n) : \Gamma_\gamma \to \mathbb{R}^n$ *a continuous function defined on the arc* Γ_γ *belonging to* γ. *Then we term the integral of the scalar product*

$$\int_\gamma f(w) \cdot dw := \sum_{j=1}^{n} \int_a^b f_j(\gamma(h))\, d\gamma_j(h)$$

as the line integral of f along the path γ. *(Here the integral on the right-hand side is an ordinary Riemann-Stieltjes integral.)*

We immediately learn from this definition that the line integral for the case $n = 1$ is an ordinary Riemann-Stieltjes integral. Before we shall discuss an example, we shall present some elementary properties of line integrals.

3.2 Theorem. *Suppose* $\gamma : [a, b] \to \mathbb{R}^n$ *is a rectifiable path,* $\gamma^- : [a, b] \to \mathbb{R}^n$ *the pertinent inverse path (cf. example 2.2 (b)),* $f, g : \Gamma_\gamma \to \mathbb{R}^n$ *continuous functions and* $c \in \mathbb{R}$. *Then*

(a) $\int_\gamma (f + g)(w) \cdot dw = \int_\gamma f(w) \cdot dw + \int_\gamma g(w) \cdot dw$;

(b) $\int_\gamma (cf)(w) \cdot dw = c \int_\gamma f(w) \cdot dw$;

(c) $\int_{\gamma^-} f(w) \cdot dw = - \int_\gamma f(w) \cdot dw$;

(d) $|\int_\gamma f(w) \cdot dw| \le (\max_{w \in \Gamma_\gamma} \|f(w)\|) \mathcal{L}(\gamma)$.

In order to give an example we shall draw on our introductory remarks and example 2.2 (a).

3.3 Example. If the commodity prices move from $(p_1^{(1)}, \dots, p_n^{(1)})$ at situation a to $(p_1^{(2)}, \dots, p_n^{(2)})$ at situation b on a rectifiable path $\gamma : [a, b] \to \mathbb{R}^n$, assuming the household income to be constant, $y = \overline{y}$, then the change in individual welfare (cf. (7)) is given by the line integral

$$\Delta W := - \int_\gamma [x_1^M(p, \overline{y}), \dots, x_n^M(p, \overline{y})] \cdot d(p_1, \dots, p_n). \tag{9}$$

Considering Roy's identity (3) and the envelope theorem (4), we obtain by use of Theorem 3.2 (b)

$$\begin{aligned} \Delta W &\overset{(3)}{=} \int_\gamma \frac{1}{\partial v / \partial y} \left(\frac{\partial v}{\partial p_1}, \dots, \frac{\partial v}{\partial p_n} \right) \cdot d(p_1, \dots, p_n) \\ &\overset{(4)}{=} \int_\gamma \frac{1}{\lambda} \left(\frac{\partial v}{\partial p_1}, \dots, \frac{\partial v}{\partial p_n} \right) \cdot d(p_1, \dots, p_n). \end{aligned} \tag{10}$$

Following Marshall (1879, 1890) we shall assume that the marginal utility of income (Lagrangean multiplier) λ is independent of the commodity prices p, so that λ is solely a function of income y: $\lambda(y)$. For instance, this case is given for a strictly separable and homothetic direct utility function, as has been assumed by Marshall[1]. Then we obtain for (10) by application of Theorem 3.2 (b)

$$\Delta W = \frac{1}{\lambda} \int_\gamma \left(\frac{\partial v}{\partial p_1}, \dots, \frac{\partial v}{\partial p_n} \right) \cdot d(p_1, \dots, p_n). \tag{11}$$

This relationship may be simplified, as we shall show soon.

At this point of our discussion we come back to the important notion of smooth paths or at least piecewise smooth paths, as they were introduced in the context of rectifiable paths. For smooth paths the following theorem guarantees the calculation of line integrals in the form of ordinary Riemann integrals.

3.4 Theorem. *If* $\gamma := (\gamma_1, \dots, \gamma_n) : [a, b] \to \mathbb{R}^n$ *is a smooth path and* $f := (f_1, \dots, f_n) : \Gamma_\gamma \to \mathbb{R}^n$ *a continuous function, then the Riemann-Stieltjes integral*

$$\int_\gamma f(w) \cdot dw = \sum_{j=1}^{n} \int_a^b f_j(\gamma(h))\gamma_j'(h)\, dh.$$

Finally, we shall indicate how line integrals may be calculated whose path of integration is composed of several path sections (cf. Theorems 2.4 and 2.6).

3.5 Theorem. *Let* $\gamma^{(k)} : [a_k, b_k] \to \mathbb{R}^n$ $(1 \leq k \leq m)$ *be rectifiable paths with* $\gamma^{(k)}(b_k) = \gamma^{(k+1)}(a_{k+1})$ $(1 \leq k \leq m-1)$. *In addition, assume* $\gamma := \gamma^{(1)} \cdots \gamma^{(m)}$ *to be the composite path and* $f : \Gamma_\gamma \to \mathbb{R}^n$ *a continuous function. Then*

$$\int_\gamma f(w) \cdot dw = \sum_{k=1}^{m} \int_{\gamma^{(k)}} f(w) \cdot dw.$$

Thus we can calculate the integral of f along γ by integrating f along the sectional paths $\gamma^{(k)}$ $(1 \leq k \leq m)$ and summing up the results.

4 Gradient Fields

With reference to Definition 3.1 the following question arises: If the points $w^{(1)}, w^{(2)} \in \mathbb{R}^n$ and $\gamma^{(1)}, \gamma^{(2)} : [a, b] \to \mathbb{R}^n$ are two rectifiable paths subject to $\gamma^{(1)}(a) = \gamma^{(2)}(a) = w^{(1)}$ and $\gamma^{(1)}(b) = \gamma^{(2)}(b) = w^{(2)}$ (i.e. $\gamma^{(1)}$ and $\gamma^{(2)}$ are two paths connecting $w^{(1)}$ and $w^{(2)}$) and if $f : \Gamma_{\gamma^{(1)}} \cup \Gamma_{\gamma^{(2)}} \to \mathbb{R}^n$ is a continuous function, in which case is valid

$$\int_{\gamma^{(1)}} f(w) \cdot dw = \int_{\gamma^{(2)}} f(w) \cdot dw\,?$$

This question refers to *path independence* of an integral. Subsequently, we shall turn to this topic.

[1] In more detail, Marshall specifies (a) the individual direct utility function to be strictly separable in the commodities, (b) the marginal utility of each commodity to be decreasing, and (c) the marginal utility of income to be constant with respect to all goods prices. Thus Marshall postulates the existence of a strictly separable and possibly homothetic utility function (cf. Ahlheim and Rose (1989), p. 38). For numerical examples cf. Takayama (1987), pp. 610–611.

We shall denote a subset $G \subset \mathbb{R}^n$ as a *region*, if G is open and if any two points out of G may be connected by a rectifiable path in G. A (continuous, differentiable, continuously differentiable) function $f : G \longrightarrow \mathbb{R}^n$ shall be named (continuous, differentiable, continuously differentiable) *vector field* on G. Correspondingly, we shall call a (continuous, differentiable, continuously differentiable) function $\varphi : G \longrightarrow \mathbb{R}$ (continuous, differentiable, continuously differentiable) *scalar field* on G.

We shall now introduce the term gradient field which plays a central role in the discussion of our problem.

4.1 Definition. *Suppose $G \subset \mathbb{R}^n$ to be a region and $f : G \longrightarrow \mathbb{R}^n$ a vector field. Then we shall call f a* gradient field *on G, if there exists a scalar field φ on G such that*

$$f(w) = (\text{grad } \varphi)(w)$$

for all $w \in G$. We shall also call φ a primitive function *belonging to f on G;* grad φ *is a gradient field with primitive function φ.*

Since the instrument of a gradient field will be of central importance for our following considerations we shall also indicate a criterion by which we may decide whether a vector field is a gradient field. For this purpose we shall refer to a theorem contributed by Schwarz.

4.2 Theorem (Schwarz). *Assume φ to be a twice continuously partially differentiable scalar field on G. Then*

$$\frac{\partial^2 \varphi}{\partial w_i \partial w_j} = \frac{\partial^2 \varphi}{\partial w_j \partial w_i}$$

for all $1 \leq i, j \leq n$.

The following criterion results from this theorem.

4.3 Criterion. *If G is convex (i.e. the straight line connecting any two points in G totally lies within G) and if $f = (f_1, \ldots, f_n) : G \longrightarrow \mathbb{R}^n$ is a continuously differentiable vector field on G, then f is a gradient field if, and only if, always for all $1 \leq i, j \leq n$*

$$\frac{\partial f_i}{\partial w_j} = \frac{\partial f_j}{\partial w_i}.$$

The importance of gradient fields will become clear by taking note of the next theorem.

4.4 Theorem. *Let $G \subset \mathbb{R}^n$ be a region and φ a continuously differentiable scalar field on G. If $w^{(1)}, w^{(2)} \in G$, then*

$$\int_\gamma (\text{grad } \varphi)(w) \cdot dw = \varphi(w^{(2)}) - \varphi(w^{(1)})$$

for each piecewise smooth path γ in G with initial point $w^{(1)}$ and endpoint $w^{(2)}$.

This theorem indicates that, assuming the existence of a gradient field, the integration related to a piecewise smooth path does not depend on the chosen path but only on the initial point and endpoint of this path. We can also show that the converse of the above theorem is true, too. The following theorem holds more precisely.

4.5 Theorem. *Suppose f is a continuous vector field with path independent line integral on the region $G \subset \mathbb{R}^n$. Let the initial point $w^{(0)} \in G$ be an arbitrary but fixed point in G, $w \in G$ be an arbitrary endpoint in G, and*

$$\varphi(w) := \int_{w^{(0)}}^{w} f(q) \cdot dq$$

where the path between $w^{(0)}$ and w is arbitrary. Then φ is a continuously differentiable scalar field on G such that

$$(\text{grad } \varphi)(w) = f(w)$$

for each $w \in G$.

We may use these results to simplify relationship (11) of Example 3.3.

4.6 Example. Continuing in Example 3.3 we at first should recall the assumption that the marginal utility of income solely depends on income: $\lambda(y)$.

If we now look at the integrand on the right-hand side of relationship (11), we discover the gradient field

$$\left(\frac{\partial v}{\partial p_1}, \dots, \frac{\partial v}{\partial p_n}\right) = \text{grad } v. \tag{12}$$

Consequently, we obtain according to Theorem 4.4

$$\Delta W = \frac{1}{\lambda(\overline{y})}[v(p_1^{(2)}, \dots, p_n^{(2)}, \overline{y}) - v(p_1^{(1)}, \dots, p_n^{(1)}, \overline{y})]. \tag{13}$$

Thus the welfare change of a household is equal to the change in utility evaluated at the inverse marginal utility of income which results from the change in prices $(p_1^{(1)}, \dots, p_n^{(1)}) \to (p_1^{(2)}, \dots, p_n^{(2)})$. The welfare change does not depend on the development of prices so that we may draw on Example 2.7. Applying Theorems 3.4 and 3.5 we obtain for relationship (13) in all

$$\begin{aligned} \Delta W &= -\sum_{j=1}^{n} \int_{p_j^{(1)}}^{p_j^{(2)}} x_j^M(p_1^{(2)}, \dots, p_{j-1}^{(2)}, h, p_{j+1}^{(1)}, \dots, p_n^{(1)}, \overline{y})\, dh \qquad (14) \\ &= \frac{1}{\lambda(\overline{y})}[v(p_1^{(2)}, \dots, p_n^{(2)}, \overline{y}) - v(p_1^{(1)}, \dots, p_n^{(1)}, \overline{y})]. \end{aligned}$$

The upper relationship of these two formulas shows that the total change in individual welfare (consumer surplus) is equal to the sum of the changes of welfare (consumer surplus) related to the commodities $1, \dots, n$; calculating the welfare change (consumer surplus) of good j $(1 \le j \le n)$ we use the final prices $p_1^{(2)}, \dots, p_{j-1}^{(2)}$ for the prices of commodities $1, \dots, j-1$ and the initial prices $p_{j+1}^{(1)}, \dots, p_n^{(1)}$ for the prices of goods $j+1, \dots, n$.

In view of the above applied assumption that the household's marginal utility of income is independent of the commodity prices, thus solely a function of income $(\lambda(y))$, we are motivated to search for the existence of alternative gradient fields in demand theory. Here is one additional, well-known case[2].

4.7 Example. We look at the household's expenditure minimization problem equivalent to the utility maximization problem (1).

$$\min_x \sum_{j=1}^{n} p_j x_j \qquad \text{subject to} \qquad u(x_1, \ldots, x_n) = \overline{U} \tag{15}$$

where $\overline{U}$ is a given level of consumer's utility.

The solution to this problem gives a set of Hicksian (compensated) demand functions $x_j^H(p, U)$ $(1 \leq j \leq n)$ and a value for the Lagrangean multiplier $\delta(p, \overline{U})$ representing the inverse marginal utility of income in the corresponding optima of the utility maximization problem and of the expenditure minimization problem ($U_{\max} = \overline{U}, y = E; E =$ expenditure): $\delta(p, \overline{U}) = 1/\lambda(p, y)$.

The pertinent expenditure function is

$$e(p, U) = \sum_{j=1}^{n} p_j x_j^H(p, U). \tag{16}$$

Due to Shephard's Lemma we have

$$x_j^H(p, U) = \frac{\partial e(p, U)}{\partial p_j} \tag{17}$$

so that $(x_1^H, \ldots, x_n^H)$ is a gradient field with reference to expenditure function e as a primitive function.

By applying Theorem 4.4 we obtain the change in the household's welfare (consumer surplus) ΔW^* that results from the changes in prices from $(p_1^{(1)}, \ldots, p_n^{(1)})$ to $(p_1^{(2)}, \ldots, p_n^{(2)})$ following a piecewise smooth path γ if the household's utility level is constant $(U = \overline{U})$.

$$\begin{aligned} \Delta W^* &= -\int_\gamma [x_1^H(p, \overline{U}), \ldots, x_n^H(p, \overline{U})] \cdot d(p_1, \ldots, p_n) \qquad (18) \\ &= e(p_1^{(1)}, \ldots, p_n^{(1)}, \overline{U}) - e(p_1^{(2)}, \ldots, p_n^{(2)}, \overline{U}). \end{aligned}$$

Thus the change in individual welfare is equal to the change in the household's expenditures originating from the commodity price changes. Here also exists path independence so that we may select the path of Example 2.7 as a special path. Considering Theorems 3.4 and 3.5 we get in all

$$\begin{aligned} \Delta W^* &= -\sum_{j=1}^{n} \int_{p_j^{(1)}}^{p_j^{(2)}} x_j^H(p_1^{(2)}, \ldots, p_{j-1}^{(2)}, h, p_{j+1}^{(1)}, \ldots, p_n^{(1)}, \overline{U})\, dh \qquad (19) \\ &= e(p_1^{(1)}, \ldots, p_n^{(1)}, \overline{U}) - e(p_1^{(2)}, \ldots, p_n^{(2)}, \overline{U}). \end{aligned}$$

As to the interpretation of (19) we refer to our comment on (14) to be analogously read.

[2] We observe that path independence in Marshallian demand theory is also given, if the marginal utility of income can be derived as a function of solely one specific commodity price. For this case of vertical Engel curves cf. Takayama (1987), pp. 610–611.

5 Concluding Remarks

Let us now further consider our main results (14) and (19) from an economic point of view, without pursuing the interrelationships of these two cases (cf. Takayama (1987), pp. 611–612).

The Marshallian measure of welfare change (14) suffers from the assumption that there exists a utility index for which the marginal utility of income is constant with respect to price changes. Samuelson (1942, pp. 81–82) has already shown that this assumption results in the restriction of unitary income elasticities of demand for all commodities. This result may easily be derived by applying Schwarz's Theorem (Theorem 4.2) often quoted to introduce path independence (cf., eg. Boadway and Bruce (1984), p. 199, and Auerbach (1985), p. 65) to the Slutsky equation belonging to the restricted Marshallian demand functions (cf. Ng (1983), pp. 94–95). In addition, Samuelson pointed out that the combined assumptions of constancy of the marginal utility of income and independence of utility of the goods imply that the elasticity of demand be always minus unity for all commodities. Reality clearly contradicts these implications.

Thus all generalized Marshallian welfare measures, although directly referring to observable "normal" demand functions are either path dependent or subject to severely restrictive assumptions. Therefore, in general, Marshallian measures must be considered as unreliable welfare indicators (see also Samuelson (1942), p. 87). However, this position may be put into a relative context as suggested by Ng (1983, pp. 95–96): "Nevertheless, for relatively small changes in prices, the income effects, even if unequal, will not be significant. Secondly, even for large changes in prices of many goods, and even if the income effects involved are not negligible or equal, they are still likely to be largely offsetting to each other in the ways they affect the measures of surplus along different paths such that, while the measure ... may be path dependent, the differences involved are likely to be small for most changes and are likely to be negligible compared to inaccuracies due to statistical and informational problems."

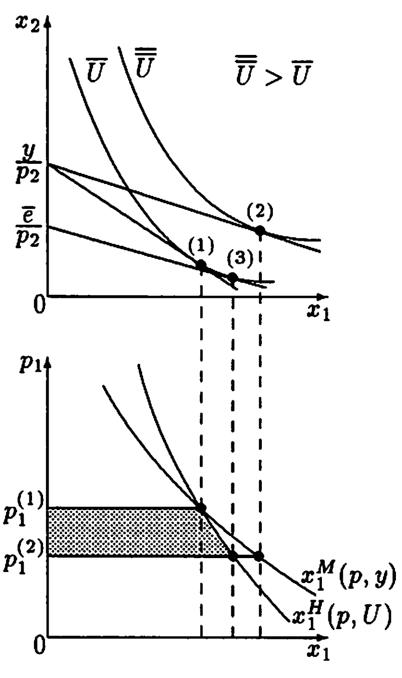

Fig. 2

The decisive theoretical breakthrough to the derivation of reliable welfare measures was accomplished by the introduction of money-metric utility measures, the equivalent variation and compensating variation of income, as proposed by Hicks (1939). Both measures of welfare change are well defined, path independent, and can be interpreted using the expenditure function. They differ only with respect to the reference prices they apply, the initial or the final prices.

In what follows we shall refer to the compensating variation (cf. also Chipman and Moore (1992)) which is given by relationship (19). The compensating variation, on the one hand, implies particular paths, on the other hand, is related to unobservable compensated demand functions.

Assume the price (p_1) of a specific commodity (x_1) to fall. Then the compensating variation is defined to be the amount of income that could be taken away from the household in situation (2) in order to maintain the same utility level as in situation (1). If the consumer is better off in (2) than in (1), the compensating variation is positive; if worse off, it is negative. This variation of income is described by Fig. 2 for the case of the two goods x_1 and x_2.

The relationship of Fig. 2 to formula (19) may be established as follows: $E = e(p_1^{(1)}, \dots, p_n^{(1)}, \overline{U}) = y$ and $e(p_1^{(2)}, \dots, p_n^{(2)}) = \overline{E}$ (belonging to $\overline{U}$). Then, with reference to the upper diagram, $\Delta W^* = y - \overline{E}$. This welfare change is depicted by the shaded area in the lower diagram. Observe: $\Delta W^* < \Delta W$. The path of the compensating variation is

$$(1) \xrightarrow{\text{no welfare change}} (3) \xrightarrow{\text{welfare change } y - \overline{E} \text{ along income expansion line}} (2).$$

If several commodity prices change, then path independence applies, i.e. we can sequentially add up welfare changes for each individual price change. The order in which we consider the price changes has no influence on the value of the total compensating variation.

However, the compensating variation and the equivalent variation as measures of welfare change also embody deficiencies which emerge in the context of the evaluation of several (household) investment projects. For instance, with respect to the compensating variation, since not only the utility levels, but also the (new) prices in the expenditure function vary, we cannot deduce the actual welfare change of the household from the sign of the difference among any two compensating variations originating from different projects (cf. Ahlheim and Rose (1989), pp. 12–18 and 67–76).

For both welfare measures, the compensating variation and the equivalent variation, the disadvantage of referring to unobservable compensated demand functions may be neglected in the meantime. Starting from Marshallian demand functions under certain conditions, money-metric utility may exactly be calculated by different approaches (cf. Rosen (1978), King (1983); McKenzie and Pearce (1982), Vartia (1983); Ahlheim and Rose (1989), pp. 59–98).

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