

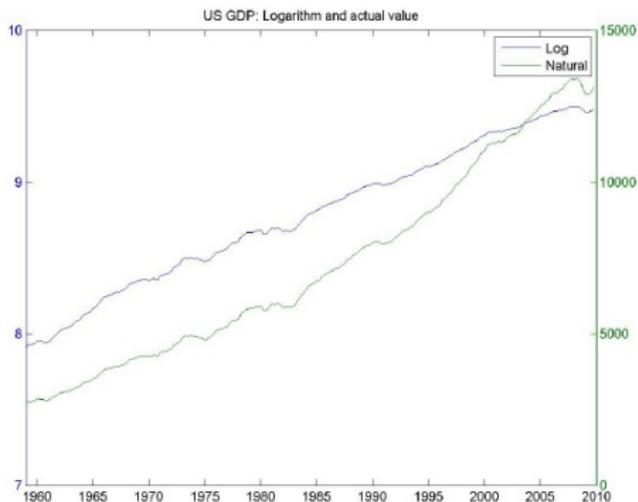
Dynamic Macroeconomics  
Problem Set 3  
Hodrick-Prescott Filter

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## 1 Introduction

- 2 Hodrick-Prescott Filter
  - Theory
  - Illustration: US GDP

## US GDP



- GDP has a clear upward trending behaviour, but there are also fluctuations around this upward trend. As we will see, this is so for many macroeconomic time series.

## Why decompose?

- Useful to decompose the original time series  $\{y_t\}_{t=1}^T$  into a trend  $\{\tau_t\}_{t=1}^T$  and cyclical component  $\{y_t^c\}_{t=1}^T$  such that

$$y_t = \tau_t + y_t^c, \quad t = 1, 2, \dots, T.$$

- Policy oriented: When is an economy in a recession phase? When in an expansion phase? Precise measurement needed to choose the right policy tools.
- Theory oriented: Often it is easier to deal with separate models for the growth and cyclical components. Many business cycle models are expressed in *stationary* variables, i.e. variables that have a constant mean of 0. To compare these models to the data, need to also de-trend the data.

## Why work with logarithms?

- Common to express variables in natural logarithms.
- It makes the graphs nicer: The vertical axis is transformed to numbers in the range of, say, 7 to 10 (which corresponds to original numbers between  $e^7 \approx 1096$  and  $e^{10} \approx 22026$ ).
- If a variable is growing exponentially, then the logarithm of the variable will grow linearly.
- $\ln y_{t+1} - \ln y_t$  is the growth rate of  $y$  since

$$\begin{aligned}\ln y_{t+1} - \ln y_t &= \ln \left( \frac{y_{t+1}}{y_t} + 1 - 1 \right) \\ &= \ln \left( 1 + \frac{y_{t+1} - y_t}{y_t} \right) \\ &= \ln (1 + g_{t+1}) \\ &\approx g_{t+1}.\end{aligned}$$

## 1 Introduction

## 2 Hodrick-Prescott Filter

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## Hodrick-Prescott Filter (HP Filter)

- Given a time-series  $\{y_t\}_{t=1}^T$  choose the trend  $\{\tau_t\}_{t=1}^T$  so that

$$\min_{\{\tau_t\}_{t=1}^T} \sum_{t=1}^T \underbrace{(y_t - \tau_t)^2}_i + \lambda \sum_{t=2}^{T-1} \underbrace{[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2}_{ii}$$

- Want to produce trend, that is *i*) close to the actual series and *ii*) does not have too large changes in the growth rate of the trend.
- $\lambda$  is a key parameter. Large values of  $\lambda$  give a smooth trend. For  $\lambda = 0$  HP Filter gives back original time-series.
- For quarterly data, usually choose  $\lambda = 1600$ .
- For annual data, usually choose  $\lambda = 100$ .

$$\lambda = 0$$

- For  $\lambda = 0$  the second objective is “switched off” and the optimisation problem becomes

$$\min_{\{\tau_t\}_{t=1}^T} \sum_{t=1}^T (y_t - \tau_t)^2.$$

- This means that we are only interested in minimizing the squared deviation between  $y_t$  and its long-run component  $\tau_t$ .
- Since we choose  $\tau_t$  in order to achieve our objective we set  $\tau_t = y_t$ .
- This means in turn that we interpret the actual time series  $y_t$  as consisting solely of a long-run component.
- At the same time we decide that there is no short-run component in the actual time series  $y_t$ .

$\lambda \rightarrow \infty$

- For  $\lambda \rightarrow \infty$  the first objective is “switched off” and the optimisation problem becomes

$$\min_{\{\tau_t\}_{t=1}^T} \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2$$

- Only interested in minimizing the squared change in the growth rate of the trend.
- One can show that this requires the trend to be some linear function of time  $\tau_t = \alpha_1 t$ , for some constant  $\alpha$ .
- Then the HP Filter would become equivalent to estimating the parameters  $\alpha_j$  in a model like

$$y_t = \alpha_0 + \alpha_1 t + \varepsilon$$

and defining the cyclical component as  $y_t^c = y_t - \hat{\alpha}_0 - \hat{\alpha}_1 t$ .

Example:  $T = 4, 0 < \lambda < \infty$

- For  $T = 4$  the minimisation problem becomes

$$\begin{aligned} \min_{\{\tau_t\}_{t=1}^4} & (y_1 - \tau_1)^2 + (y_2 - \tau_2)^2 + (y_3 - \tau_3)^2 + (y_4 - \tau_4)^2 \\ & + \lambda[(\tau_3 - \tau_2) - (\tau_2 - \tau_1)]^2 + \lambda[(\tau_4 - \tau_3) - (\tau_3 - \tau_2)]^2 \end{aligned}$$

- With the first-order conditions

$$\text{w.r.t } \tau_1 : -2(y_1 - \tau_1) + 2\lambda[(\tau_3 - \tau_2) - (\tau_2 - \tau_1)] = 0$$

$$\text{w.r.t } \tau_2 : -2(y_2 - \tau_2) - 4\lambda[(\tau_3 - \tau_2) - (\tau_2 - \tau_1)] + 2\lambda[(\tau_3 - \tau_2) - (\tau_2 - \tau_1)] = 0$$

$$\text{w.r.t } \tau_3 : -2(y_3 - \tau_3) - 4\lambda[(\tau_4 - \tau_3) - (\tau_3 - \tau_2)] + 2\lambda[(\tau_3 - \tau_2) - (\tau_2 - \tau_1)] = 0$$

$$\text{w.r.t } \tau_4 : -2(y_4 - \tau_4) + 2\lambda[(\tau_4 - \tau_3) - (\tau_3 - \tau_2)] = 0$$

Example:  $T = 4$

- They can be written in matrix notation as

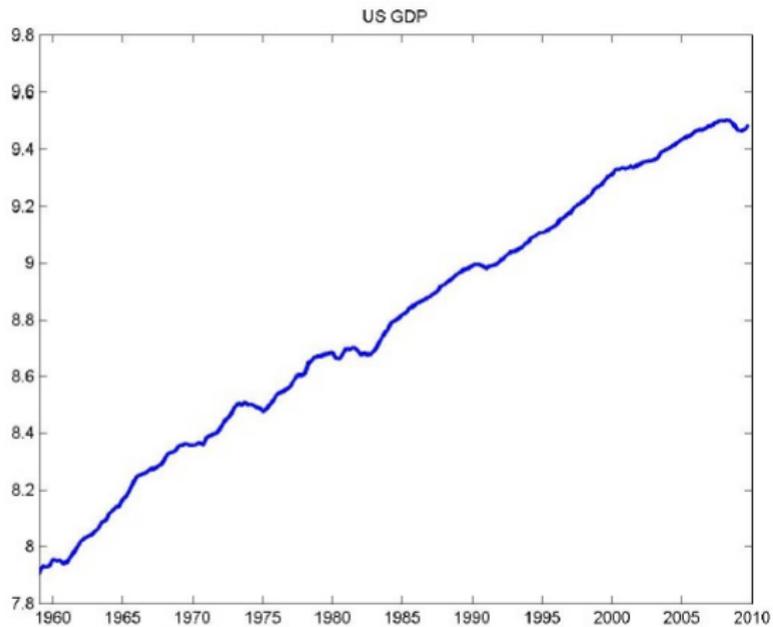
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \lambda + 1 & -2\lambda & \lambda & 0 \\ -2\lambda & 5\lambda + 1 & -4\lambda & \lambda \\ \lambda & -4\lambda & 5\lambda + 1 & -2\lambda \\ 0 & \lambda & -2\lambda & \lambda + 1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix}$$

- and be solved by matrix inversion to get

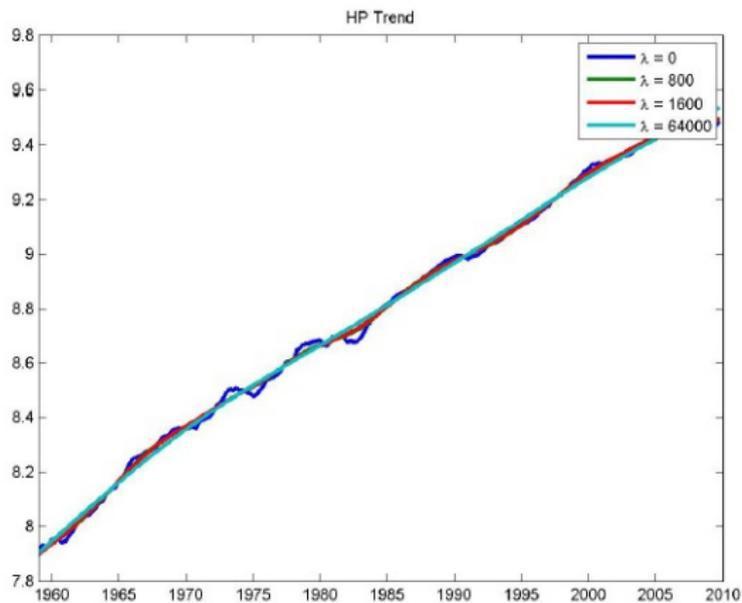
$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} \lambda + 1 & -2\lambda & \lambda & 0 \\ -2\lambda & 5\lambda + 1 & -4\lambda & \lambda \\ \lambda & -4\lambda & 5\lambda + 1 & -2\lambda \\ 0 & \lambda & -2\lambda & \lambda + 1 \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

Since  $\lambda$  is chosen and  $y$  is the original data,  $\tau$  can be computed.

## Original data

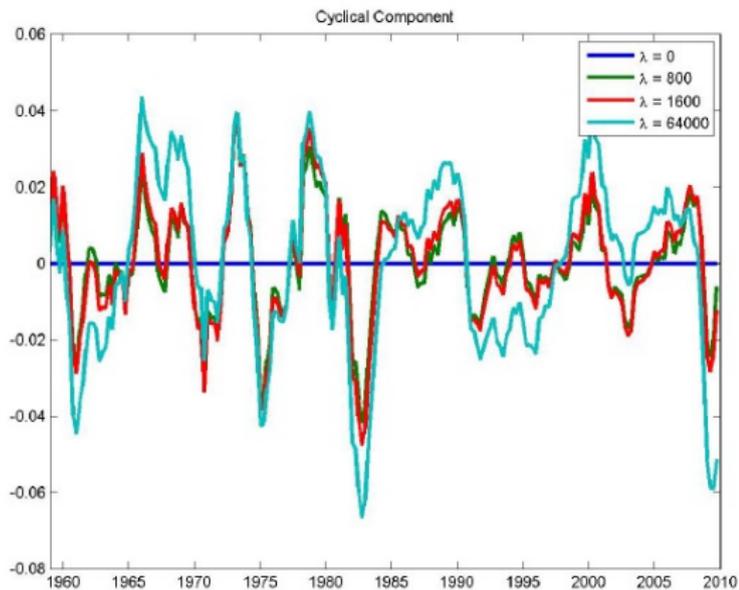


## Trend - Influence of $\lambda$



Higher is  $\lambda$  - smother is a trend

## Cycle - Influence of $\lambda$



With higher  $\lambda$  more cyclical components are excluded from the original data.