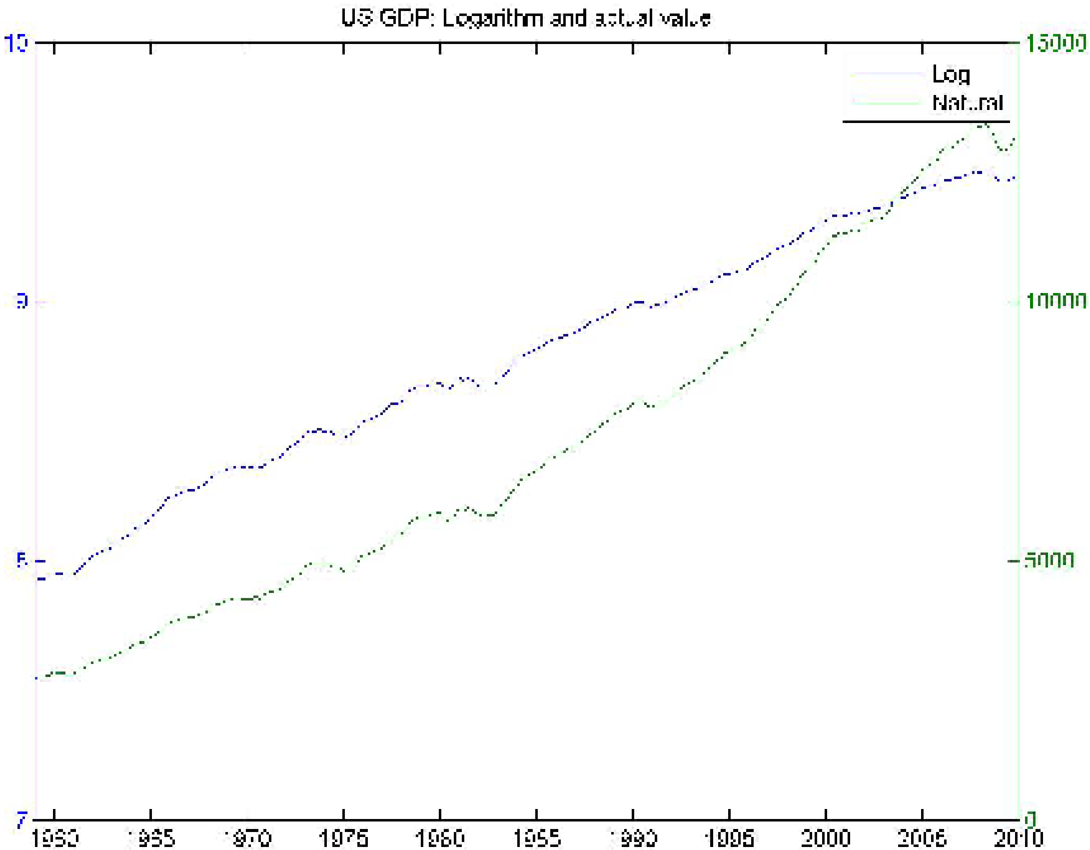
Dynamic Macroeconomics: Problem Set 3 Hodrick-Prescott Filter

Universität Siegen

US GDP



- GDP has a clear upward trending behaviour, but there are also fluctuations around this upward trend. As we will see, this is so for many macroeconomic time series.

Why decompose?

- Useful to decompose the original time series $\{y_t\}_{t=1}^{T}$ into a trend $\{\tau_t\}_{t=1}^{T}$ and cyclical component $\{y_t^c\}_{t=1}^{T}$ such that

$$y_t = \tau_t + y_t^c, \quad t = 1, 2, \cdots, T.$$

- Policy oriented: When is an economy in a recession phase? When in an expansion phase? Precise measurement needed to choose the right policy tools.
- Theory oriented: Often it is easier to deal with separate models for the growth and cyclical components. Many business cycle models are expressed in *stationary* variables, i.e. variables that have a constant mean of 0. To compare these models to the data, need to also de-trend the data.

Why work with logarithms?

- Common to express variables in natural logarithms.
- It makes the graphs nicer: The vertical axis is transformed to numbers in the range of, say, 7 to 10 (which corresponds to original numbers between $e^7 \approx 1096$ and $e^{10} \approx 22026$).
- If a variable is growing exponentially, then the logarithm of the variable will grow linearly.
- $\ln y_{t+1} - \ln y_t$ is the growth rate of y since

$$
\begin{aligned}
\ln y_{t+1} - \ln y_t = &= \ln \left(\frac{y_{t+1}}{y_t} + 1 - 1 \right) \\
&= \ln \left(1 + \frac{y_{t+1} - y_t}{y_t} \right) \\
&= \ln \left(1 + g_{t+1} \right) \\
&\approx g_{t+1}.
\end{aligned}
$$

Hodrick-Prescott Filter (HP Filter)

- Given a time-series $\{y_t\}_{t=1}^{T}$ choose the trend $\{\tau_t\}_{t=1}^{T}$ so that

$$\min_{\{\tau_t\}_{t=1}^{T}} \sum_{t=1}^{T} \underbrace{(y_t - \tau_t)^2}_{i} + \lambda \sum_{t=2}^{T-1} \underbrace{[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2}_{ii}$$

- Want to produce trend, that is *i)* close to the actual series and *ii)* does not have too large changes in the growth rate of the trend.
- λ is a key parameter. Large values of λ give a smooth trend. For $\lambda = 0$ HP Filter gives back original time-series.
- For quarterly data, usually choose $\lambda = 1600$.
- For annual data, usually choose $\lambda = 100$.

$\lambda = 0$

- For $\lambda = 0$ the second objective is "switched off" and the optimisation problem becomes

$$\min_{\{\tau_t\}_{t=1}^{T}} \sum_{t=1}^{T} (y_t - \tau_t)^2.$$

- This means that we are only interested in minimizing the squared deviation between y_t and its long-run component τ_t.
- Since we choose τ_t in order to achieve our objective we set $\tau_t = y_t$.
- This means in turn that we interpret the actual time series y_t as consisting solely of a long-run component.
- At the same time we decide that there is no short-run component in the actual time series y_t.

$\lambda \to \infty$

- For $\lambda \to \infty$ the first objective is "switched off" and the optimisation problem becomes

$$\min_{\{\tau_t\}_{t=1}^{T}} \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2$$

- Only interested in minimizing the squared change in the growth rate of the trend.
- One can show that this requires the trend to be some linear function of time $\tau_t = \alpha_1 t$, for some constant α.
- Then the HP Filter would become equivalent to estimating the parameters α_i in a model like

$$y_t = \alpha_0 + \alpha_1 t + \varepsilon$$

and defining the cyclical component as $y_t^c = y_t - \hat{\alpha}_0 - \hat{\alpha}_1 t$.

Example: $T = 4, 0 < \lambda < \infty$

- For $T = 4$ the minimisation problem becomes

$$\min_{\{\tau_t\}_{t=1}^{4}} (y_1 - \tau_1)^2 + (y_2 - \tau_2)^2 + (y_3 - \tau_3)^2 + (y_4 - \tau_4)^2$$
$$+ \lambda[(\tau_3 - \tau_2) - (\tau_2 - \tau_1)]^2 + \lambda[(\tau_4 - \tau_3) - (\tau_3 - \tau_2)]^2$$

- With the first-order conditions

$$\begin{aligned}
&\text{w.r.t } \tau_1 : -2(y_1 - \tau_1) + 2\lambda[(\tau_3 - \tau_2) - (\tau_2 - \tau_1)] &&= 0\\
&\text{w.r.t } \tau_2 : -2(y_2 - \tau_2) - 4\lambda[(\tau_3 - \tau_2) - (\tau_2 - \tau_1)] + 2\lambda[(\tau_3 - \tau_2) - (\tau_2 - \tau_1)] &&= 0\\
&\text{w.r.t } \tau_3 : -2(y_3 - \tau_3) - 4\lambda[(\tau_4 - \tau_3) - (\tau_3 - \tau_2)] + 2\lambda[(\tau_3 - \tau_2) - (\tau_2 - \tau_1)] &&= 0\\
&\text{w.r.t } \tau_4 : -2(y_4 - \tau_4) + 2\lambda[(\tau_4 - \tau_3) - (\tau_3 - \tau_2)] &&= 0
\end{aligned}$$

Example: $T = 4$

- They can be written in matrix notation as

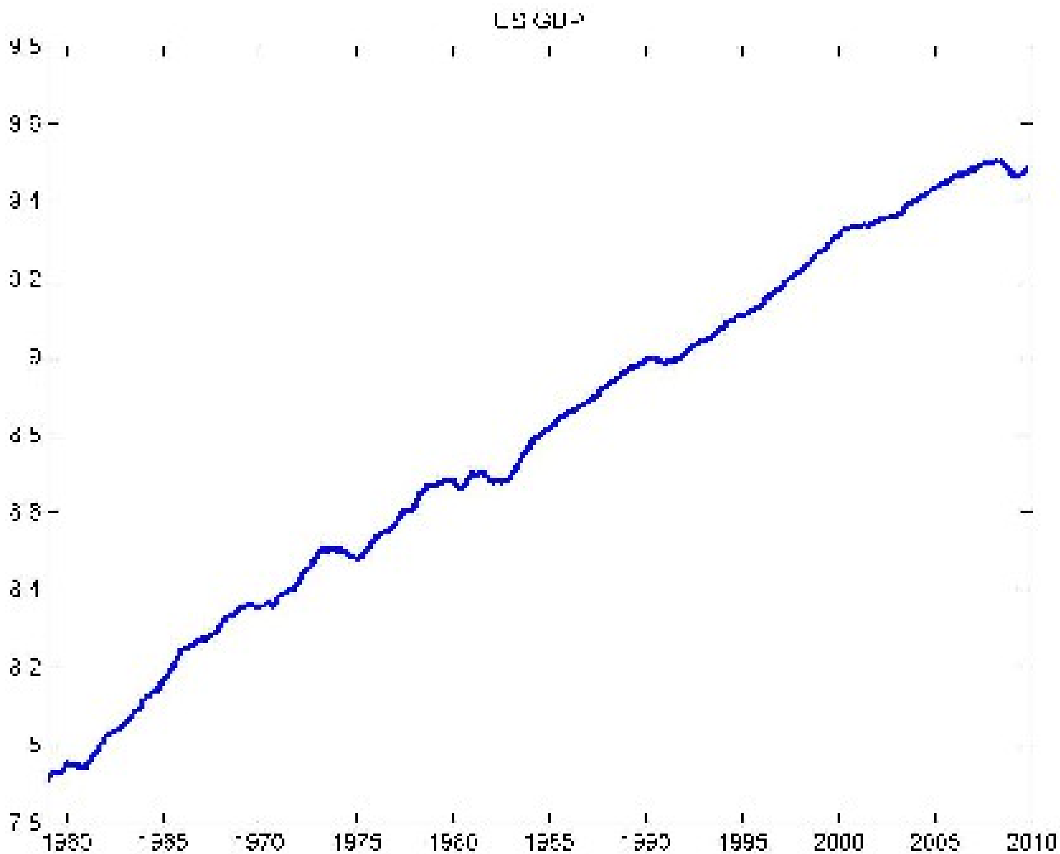
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \lambda+1 & -2\lambda & \lambda & 0 \\ -2\lambda & 5\lambda+1 & -4\lambda & \lambda \\ \lambda & -4\lambda & 5\lambda+1 & -2\lambda \\ 0 & \lambda & -2\lambda & \lambda+1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix}$$

- and be solved by matrix inversion to get

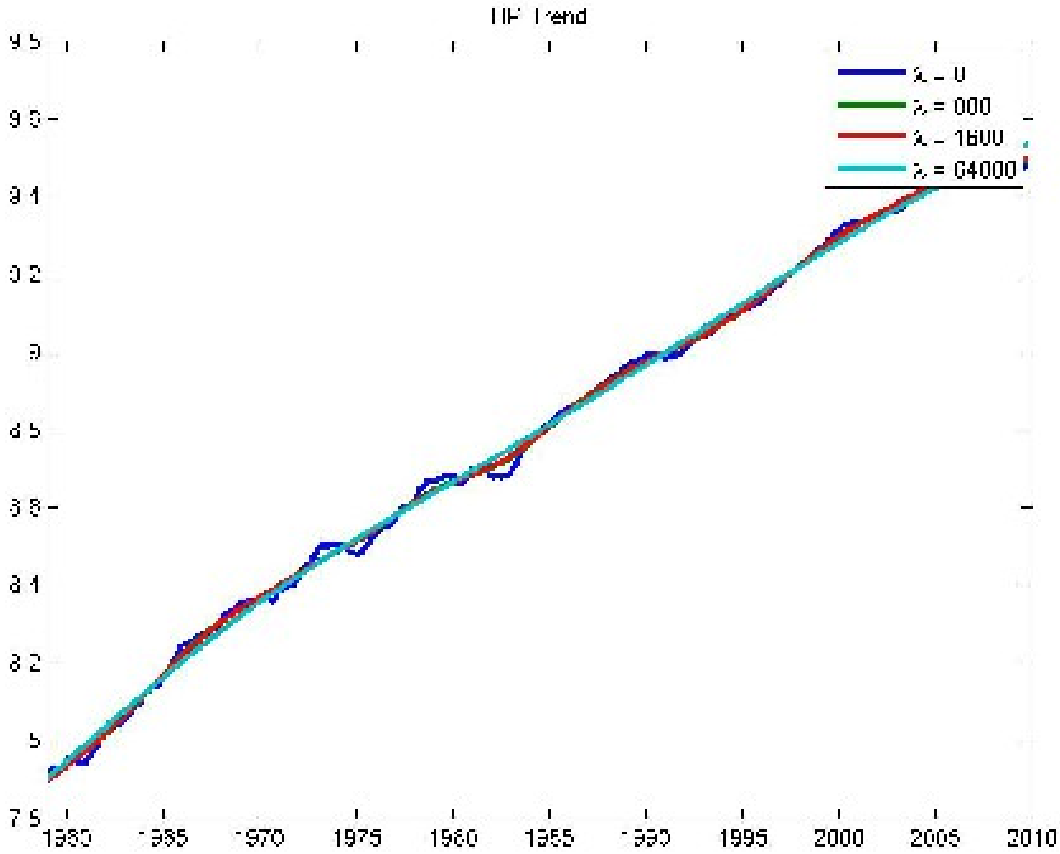
$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} \lambda+1 & -2\lambda & \lambda & 0 \\ -2\lambda & 5\lambda+1 & -4\lambda & \lambda \\ \lambda & -4\lambda & 5\lambda+1 & -2\lambda \\ 0 & \lambda & -2\lambda & \lambda+1 \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

Since λ is chosen and y is the original data, τ can be computed.

Original data



Trend - Influence of λ



Cycle - Influence of λ