Dynamic Macroeconomics

Problem Set 3

1. Assume that we have an economy that can be described as follows:

    - **Households and preferences**
        - The economy is inhabited by one single representative consumer.
        - This consumer has preferences over an infinite stream of consumption $c_0, c_1, ... = \{c_t\}_{t=0}^{\infty}$.
        - The consumer's lifetime utility function is assumed to be **time-separable** and given by:

        $$U(\{c\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

        - β is the individuals subjective time discount factor. We assume that $0 < \beta < 1$ holds.
        - $u\left(c_t\right)$ denotes the period utility function. We assume that it is strictly increasing and concave and is given by the following function:

        $$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}.$$

        We assume that $\sigma \geq 0$ (a risk aversion parameter).
    - **Production**
        - The production technology of the economy is given by:

        $$y = f(k),$$

        where y denotes the output, k the used capital and $f(.)$ is a production function.
        - We further assume that $f(k)$ is Cobb-Douglas and thus given by:

        $$f(k) = Ak^{\alpha}.$$

        A denotes the level of 'knowledge'. Assume that $0 < \alpha < 1$.
        - We further assume that the consumer owns the production technology, i.e. there are no firms.

a) Derive the budget constraint of the representative consumer.

Solution: *The budget constraint of the representative consumer is that his consumption, c_t, and investment, i_t, have to be equal to total production, y_t, i.e.*

$$c_t + i_t = y_t \equiv f(k_t).$$

Combining this with the equation for the evolution of the capital stock

$$k_{t+1} = (1-\delta)k_t + i_t,$$

where δ is the depreciation rate (capturing effects like the destruction of capital goods during production), we get that

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

which is the equation we will use in the following. Note that we can interpret the right-hand side of this equation as the total resources available after the production process in period t has ended: Consisting of the newly produced goods $f(k_t)$ and the remaining capital stock $(1-\delta)k_t$, which survived the production process.

b) Setup the inter-temporal optimization problem of the individual.

Solution: *The optimization problem is to choose sequences of capital and consumption $\{k_{t+1}, c_t\}_{t=0}^{\infty}$ which maximize the life-time utility function and satisfies in each period the budget constraint. Additionally, the transversality condition has to be satisfied. Formally the problem is*

$$\begin{aligned}
&\max_{\{k_{t+1},c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&\textit{such that} \\
&c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \\
&\lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0 \\
&k_0 \textit{ given}
\end{aligned}$$

The exogenous variables/parameters of the model are the initial capital stock k_0, the utility parameters β and σ, the production function parameters α and A and the depreciation rate δ. The endogenous variables are c_t, k_{t+1} for $t = 0, 1, 2, \ldots$. We are thus looking for solutions of the form $c_t = h(k_0, \beta, \sigma, \alpha, A, \delta)$ and $k_{t+1} = g(k_0, \beta, \sigma, \alpha, A, \delta)$, where $h()$ and $g()$ are the solution functions we are looking for.

c) Setup the Lagrange function associated with the inter-temporal optimization problem and show that the first-order conditions of the optimization problem can be written as follows:

$$u'(c_t) = \beta u'(c_{t+1}) \left[1 + f'(k_{t+1}) - \delta\right] \tag{1}$$

and

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \tag{2}$$

c) Setup the Lagrange function associated with the inter-temporal optimization problem and show that the first-order conditions of the optimization problem can be written as follows:

$$u'(c_t) = \beta u'(c_{t+1}) \left[1 + f'(k_{t+1}) - \delta\right] \tag{3}$$

and

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \tag{4}$$

Solution: *The Lagrange function is built up with help of the objective function (here - the life-time utility) and the given constraints (here - the budget constraint). Think about the budget constraint: it has to be 0 but suppose it was positive. The right-hand side is total resources available after the production process. The left-hand side determines how these resources should be used - consumed or invested. If we substract from the RHS the LHS and assume that it is positive, then it gives us unspent resources which provide the additional utility (that's why there is "+" in front of the budget constraint in the Lagrangian function). But these resources are valued in some currency, while the utility is measured in utility units. So, we are not allowed to simply add them together. The Lagrange multiplier converts the dollars into units and therefore is measured in utils per dollar. The Lagrange function for the optimization problem is given by*

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\right]. \tag{5}$$

The second term can be interpreted as the utility of unspent resources, while the Lagrangian multiplier is the marginal utility of unspent resources (marginal value of relaxing the constraint). We have to take first order conditions with respect to c_t, k_{t+1} and the Lagrange multiplier λ_t for all values of $t = 0, 1, 2, \ldots, 10, 11, \ldots$. To do this, it is *helpful to write the summation signs in* (5) *a bit more explicitly. Doing this for periods* $t = 10$ *and* $t = 11$ *we get*

$$\begin{aligned}\mathcal{L} = \ldots + \beta^{10}u(c_{10}) + \lambda_{10}\left[f(k_{10}) + (1-\delta)k_{10} - c_{10} - k_{11}\right] \\ +\beta^{11}u(c_{11}) + \lambda_{11}\left[f(k_{11}) + (1-\delta)k_{11} - c_{11} - k_{12}\right] + \ldots\end{aligned}$$

Now we can obtain the first order conditions with respect to $c_{10}, k_{11}, \lambda_{10}$, *i.e. the choice variables the consumer has to choose in period* $t = 10$, *and also with respect to* c_{11} *and* λ_{11} *(I do not consider the derivative w.r.t.* k_{12}, *because I do not need it). They are given by*

$$\frac{\partial \mathcal{L}}{\partial c_{10}} = 0 \qquad \leftrightarrow \beta^{10}u'(c_{10}) - \lambda_{10} = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial k_{11}} = 0 \qquad \leftrightarrow -\lambda_{10} + \lambda_{11}(f'(k_{11}) + 1 - \delta) = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{10}} = 0 \qquad \leftrightarrow f(k_{10}) + (1-\delta)k_{10} - c_{10} - k_{11} = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial c_{11}} = 0 \qquad \leftrightarrow \beta^{11}u'(c_{11}) - \lambda_{11} = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{11}} = 0 \qquad \leftrightarrow f(k_{11}) + (1-\delta)k_{11} - c_{11} - k_{12} = 0 \tag{10}$$

Using (6) *and* (9) *in* (7), *we get*

$$\begin{aligned}-\lambda_{10} + \lambda_{11}(f'(k_{11}) + 1 - \delta) &= 0 \\ -\beta^{10}u'(c_{10}) + \beta^{11}u'(c_{11})(f'(k_{11}) + 1 - \delta) &= 0 \\ \beta^{10}u'(c_{10}) &= \beta^{11}u'(c_{11})(f'(k_{11}) + 1 - \delta) \\ u'(c_{10}) &= \beta u'(c_{11})(f'(k_{11}) + 1 - \delta)\end{aligned}$$

our usual Euler equation. Since the derivation would hold for all other values for t *as well, we can write more generally that*

$$u'(c_t) = \beta u'(c_{t+1})(f'(k_{t+1}) + 1 - \delta) \tag{11}$$

So, in the optimum, the utility loss from saving more today must be equal to the discounted utility gain tomorrow. The total differential of the ife-time utility is equal to 0. In addition to the Euler equation, the budget constraint has to be satisfied as well (given by (8)*). If we now use the given functional forms for* $u()$ *and* $f()$, *we get (making use of* $u'(c_t) = c_t^{-\sigma}$ *and* $f'(k_t) = \alpha A k_t^{\alpha-1}$*)*

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}(1 + \alpha A k_{t+1}^{\alpha-1} - \delta) \tag{12}$$

$$c_t + k_{t+1} = A k_t^{\alpha} + (1-\delta)k_t. \tag{13}$$

d) Compute the models steady state values of capital k^* and consumption c^*. Discuss how the steady state values depend upon the parameters of the model.

Solution: *Equations* (12) *and* (13) *are a system of non-linear difference equations in* c_t *and* k_t. *(Difference equation simply means that variables at different times appear in the equations (here at time t and time* $t+1$*). Ideally, we would want to solve this equation system to get expressions like* $c_t = h(k_0, \beta, \sigma, \alpha, A, \delta)$ *and* $k_{t+1} = g(k_0, \beta, \sigma, \alpha, A, \delta)$. *However, this is usually not doable form a mathematical perspective. Therefore we have to look at simplified version of* (12) *and* (13) *to get solutions. These simplifications come in two forms: Either we assume that the endogenous variables take on constant values over time (are in their steady state) or we make assumptions about the paramter values, that make* (12) *and* (13) *simpler). This question is about the steady state case. In the steady state the values of capital and consumption are constant, i.e.,* $k_{t+1} = k_t = k^*$ *and* $c_{t+1} = c_t = c^*$. *From the Euler equation we then get*

$$
\begin{aligned}
(c^*)^{-\sigma} &= \beta(c^*)^{-\sigma}(1 + \alpha A(k^*)^{\alpha-1} - \delta) \\
1 &= \beta(1 + \alpha A(k^*)^{\alpha-1} - \delta) \\
\frac{1}{\beta} - 1 + \delta &= \alpha A(k^*)^{\alpha-1} \\
\frac{1}{\alpha A}\left(\frac{1}{\beta} - 1 + \delta\right) &= (k^*)^{\alpha-1} \\
\left[\frac{1}{\alpha A}\left(\frac{1}{\beta} - (1-\delta)\right)\right]^{\frac{1}{\alpha-1}} &= k^* \\
\left[\frac{1-\beta(1-\delta)}{\alpha\beta A}\right]^{\frac{1}{\alpha-1}} &= k^* \\
\left[\frac{\alpha\beta A}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}} &= k^*
\end{aligned}
$$

Defining $\beta = \frac{1}{1+\rho}$ we can simplify this further to

$$\left[\frac{\alpha\beta A}{1-\frac{1}{1+\rho}(1-\delta)}\right]^{\frac{1}{1-\alpha}} = k^*$$

$$\left[\frac{\alpha\beta A}{\frac{1+\rho}{1+\rho}-\frac{1-\delta}{1+\rho}}\right]^{\frac{1}{1-\alpha}} = k^*$$

$$\left[\frac{\alpha\beta A}{\frac{\rho+\delta}{1+\rho}}\right]^{\frac{1}{1-\alpha}} = k^*$$

$$\left[\frac{\alpha\beta A}{\beta(\rho+\delta)}\right]^{\frac{1}{1-\alpha}} = k^*$$

$$\left[\frac{\alpha A}{(\rho+\delta)}\right]^{\frac{1}{1-\alpha}} = k^*$$

From the budget constraint we get that $c^* = f(k^*) - \delta k^*$. We see that the steady-state capital stock does depend upon the parameters of the utility and production function (but not on the initial capital stock k_0). The best way to interpret how these parameters influence the steady state capital stock, is to think about benefits and costs of investing (since the capital stock is nothing but accumulated investment). Investment should be higher, if it is very profitable. In our setup, the profitability of investment is governed by the parameters α, A, δ. α is the elasticity of final production with respect to the available capital stock. If it is high, this means that each additional unit of capital leads to a large increase in output (i.e. has a high marginal product). Therefore a high α is associated with a high k^*. A is a productivity parameter. If it is high, then each additional unit of capital leads to a large increase in output (i.e. has a high marginal product). Therefore a high A is associated with a high k^*. δ on the other hand has a negative impact on k^*. If δ is high, then a huge part of your invested capital is destroyed during the production process and therefore the overall return on investment is lower. Finally, the preference parameter ρ has a negative relationship with k^*. High values of ρ indicate a very impatient household (i.e. he likes to consume now, rather than save).

From now one assume that $\sigma = 1$ (so that the utility function becomes $u(c_t) = \ln c_t$) and that $\delta = 1$. Also assume that $\alpha = 0.33, \beta = 0.97, A = 1$.

e) Show that the optimal solution for the evolution of the capital stock is given by

$$k_{t+1} = \alpha\beta k_t^\alpha.$$

Solution: *If $k_{t+1} = \alpha\beta k_t^\alpha$ is a solution than it has to satisfy the Euler equation and the budget constraint, which are given by (using the assumption on the parameter values)*

$$\begin{aligned}
\frac{1}{c_t} &= \beta\frac{1}{c_{t+1}}\alpha k_{t+1}^{\alpha-1} \\
c_t + k_{t+1} &= k_t^\alpha.
\end{aligned}$$

From the budget constraint we get

$$c_t = k_t^\alpha - k_{t+1} = k_t^\alpha - \alpha\beta k_t^\alpha = (1-\alpha\beta)k_t^\alpha,$$

where the second equality use the proposed solution for k_{t+1}. Rewriting the Euler equation gives

$$\begin{aligned}
\frac{c_{t+1}}{c_t} &= \beta\alpha k_{t+1}^{\alpha-1} \\
\frac{(1-\alpha\beta)k_{t+1}^\alpha}{1-\alpha\beta)k_t^\alpha} &= \beta\alpha k_{t+1}^{\alpha-1} \\
\frac{k_{t+1}^\alpha}{k_t^\alpha} &= \beta\alpha k_{t+1}^{\alpha-1} \\
\frac{k_{t+1}^\alpha}{k_{t+1}^{\alpha-1}} &= \beta\alpha k_t^\alpha \\
k_{t+1} &= \alpha\beta k_t^\alpha \\
\alpha\beta k_t^\alpha &= \alpha\beta k_t^\alpha
\end{aligned}$$

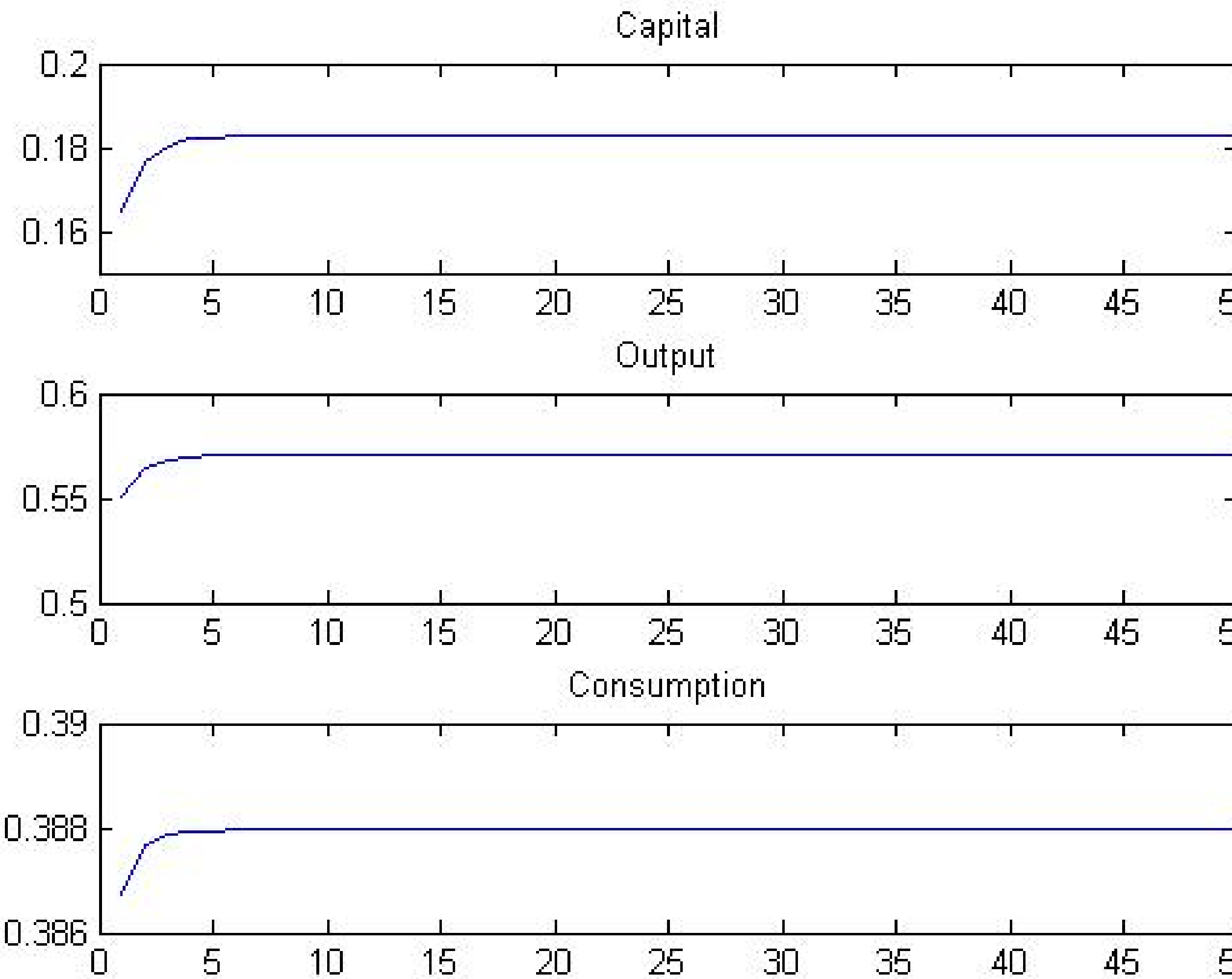
which is a true expression. Hence $k_{t+1} = \alpha\beta k_t^\alpha = \alpha\beta y$ does satisfy Euler equation and budget constraint and thus is the solution to the optimization problem.
Note that this equation allows one to simulate time paths of the economy. We are given the initial capital stock k_0. This allows us to compute $y_0 = k_0^\alpha$. Then we can compute $k_1 = \alpha\beta y_0$ and $c_0 = y_0 - k_1$. Then we go to period 1. We know, from period 0, the value k_1. This allows us to compute $y_1 = k_1^\alpha$. Then we can compute $k_2 = \alpha\beta y_1$ and $c_1 = y_1 - k_2$. Then we go to period 2, and so on. (You should not do this by hand. One can easily implement this computation in Excel and/or MatLab). In this way, we are able to analyze, for example the adjustment process after some of the parameter values have changed - see questions f) and

g) for two examples.
Note that in this case the fraction of income saved (which is equal to the fraction invested) is given by $\frac{i_t}{y_t} = \frac{\alpha\beta k_t^\alpha}{k_t^\alpha} = \alpha\beta$ *is constant (and thus does not depend upon the interest rate/marginal product of capital). This result depends upon the assumption* $u(c) = \ln c$*, because for this utility function income and substitution effects offset each other, so that the saving rate becomes independent of the interest rate (as was the case in the two-period model form PS2). In case if* $\sigma \prec 1$ *(a risk averse consumer, i.e. does not like a risk and prefers stability), a substitution effect dominates, and hence, investments increase in order to be able to consume more in next periods. In case of* $\sigma \succ 1$ *(a risk-loving consumer), an income effect dominates, and investments decrease.*

f) Use the results from part e) to simulate the convergence process of the economy if it starts from a initial level of capital that is ten-percent below its steady state level.

Solution: *If* $k_0 \prec k^*$*, we are at the south-west region of the phase diagram. The capital stock is so low, the its marginal product is high. Then, low consumption means that* $k \uparrow \Rightarrow y_t \uparrow \Rightarrow c_t$*. The economy adjusts along the saddle path.*

Figure 1: Starting with $k_0 = 0.9k^*$



g) Use the results from part e) to simulate the convergence process of the economy if the discount rate increases from $\beta = 0.97$ to $\beta = 0.98$.

Solution: *The change of β leads to a shift of $\Delta c_t = 0$ because this variable is not present on the axis, and the underlie equation depends on β. Shifting $\Delta c_t = 0$ means a new long-run equilibrium (new steady state values). Higher β implies that a consumer cares more about future. Then, higher savings will be. Since k_0 is given, it takes one period before it can change: increasing investment today, the following changes will be seen in the next period.*