Dynamic Macroeconomics

Problem Set 3

We showed that the equilibrium of the economy is described by the equations

$$\begin{aligned} c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma}(1 + \alpha A k_{t+1}^{\alpha-1} - \delta) \\ c_t + k_{t+1} &= A k_t^{\alpha} + (1-\delta)k_t. \end{aligned}$$

For simplicity, we now assume that $A = 1$ and $\sigma = 1$ (i.e. the utility function is $u(c) = \ln c$.[1] Then the above equations become

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}}(1 + \alpha k_{t+1}^{\alpha-1} - \delta) \tag{1}$$

$$c_t + k_{t+1} = k_t^{\alpha} + (1-\delta)k_t. \tag{2}$$

We can rewrite equation (1) as

$$\begin{aligned} \frac{1}{c_t} &= \beta \frac{1}{c_{t+1}}(1 + \alpha k_{t+1}^{\alpha-1} - \delta) \\ c_{t+1} &= \beta c_t(1 + \alpha k_{t+1}^{\alpha-1} - \delta) \\ c_{t+1} - c_t &= \beta c_t(1 + \alpha k_{t+1}^{\alpha-1} - \delta) - c_t \\ c_{t+1} - c_t &= c_t\left[\beta(1 + \alpha k_{t+1}^{\alpha-1} - \delta) - 1\right]. \end{aligned} \tag{3}$$

We can rewrite equation (2) as

$$\begin{aligned} c_t + k_{t+1} &= k_t^{\alpha} + (1-\delta)k_t \\ k_{t+1} - k_t &= k_t^{\alpha} - \delta k_t - c_t. \end{aligned} \tag{4}$$

From (3) we derive the steady-state capital stock (by setting $c_{t+1} - c_t = 0$). It is given by

$$k^* = \left[\frac{\alpha}{(1/\beta) - 1 + \delta}\right]^{\frac{1}{1-\alpha}}. \tag{5}$$

If this value of the capital stock is obtained, then consumption does not change over time, i.e. $c_{t+1} = c_t = c^*$. For later use, fix $\beta = 0.95, \alpha = 0.4, \delta = 0.2$, which results in $k^* = 2.15$.

We now turn to the dynamic behavior of consumption over time. This depends upon the relation between the initial capital stock k_0 and the steady-state capital stock k^*:

[1] For the general case, refer to the lecture slides.

Suppose that in the initial period $k_0 > k^*$. Then $\alpha k_0^{\alpha-1} = \alpha \frac{1}{k_0^{1-\alpha}} < \alpha \frac{1}{(k^*)^{1-\alpha}}$ so that the right hand side of (3) is negative and consumption decreases over time. For example, suppose $k_0 = 4 > k^* = 2.15$. Then $\beta(\alpha k_0^{\alpha-1} + 1 - \delta) - 1 = -0.074 < 0$. Hence consumption is decreasing over time.

Suppose that in the initial period $k_0 < k^*$. Then $\alpha k_0^{\alpha-1} = \alpha \frac{1}{k_0^{1-\alpha}} > \alpha \frac{1}{(k^*)^{1-\alpha}}$ so that the right hand side of (3) is positive and consumption increase over time. For example, suppose $k_0 = 1 < k^* = 2.15$. Then $\beta(\alpha k_0^{\alpha-1} + 1 - \delta) - 1 = 0.14 < 0$. Hence consumption is increasing over time.

Suppose that in the initial period $k_0 = k^*$. Then $\alpha k_0^{\alpha-1} = \alpha \frac{1}{k_0^{1-\alpha}} = \alpha \frac{1}{(k^*)^{1-\alpha}}$ so that the right hand side of (3) is zero and consumption stays constant over time. For example, suppose $k_0 = 2.15 = k^* = 2.15$. Then $\beta(\alpha k_0^{\alpha-1} + 1 - \delta) - 1 = 0$. Hence consumption is constant over time.

This gives us our first building block of the phase diagram:

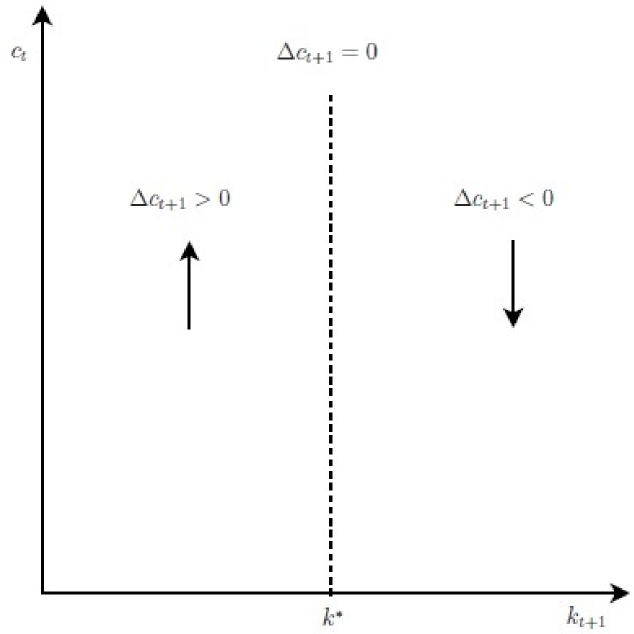

Figure 1: Δc_{t+1} diagram

We now turn the dynamics of the capital stock over time. From (4) we get, by imposing $k_{t+1} = k_t = k^*$ that

$$c^* = (k^*)^\alpha - \delta k^*. \tag{6}$$

If $c^* > (k^*)^\alpha - \delta k^*$, then the left-hand side of (4) will be negative and hence $k_{t+1} - k_t < 0$, i.e. the capital stock will decrease. If $c^* < (k^*)^\alpha - \delta k^*$, then the left-hand side of (4) will be positive and hence $k_{t+1} - k_t > 0$, i.e. the capital stock will increase. If $c^* = (k^*)^\alpha - \delta k^*$, then the left-hand side of (4) will be zero and hence $k_{t+1} - k_t = 0$, i.e. the capital stock stays constant.

This gives us our second building block of the phase diagram:

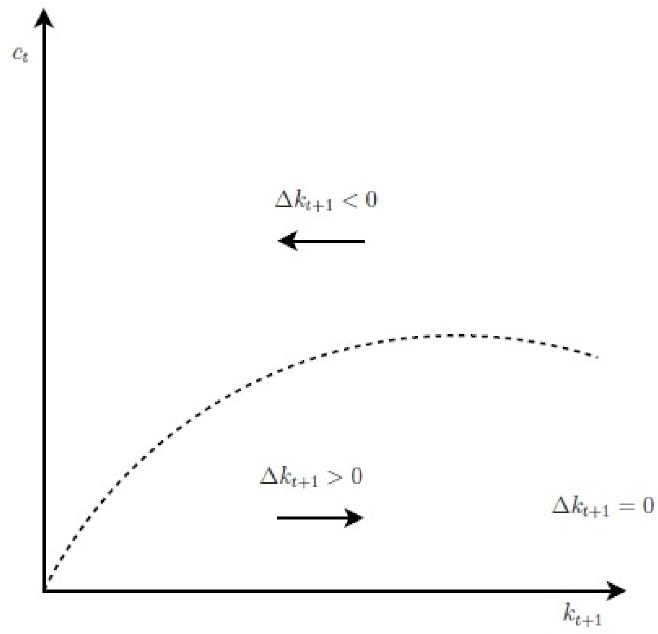

Figure 2: Δk_{t+1} diagram

Combining the two figures we get the full phase diagram 4, which also indicates the convergence behavior towards the steady state. Only points on the saddle path SS are attainable. This is not as restrictive as it may seem, as the location of the saddle path is determined by the economy, i.e. the parameters of the model, and could in principle be in an infinite number of places depending on the particular values of the parameters. The arrows denote the dynamic behavior of c_t and k_t. This depends on which of four possible regions the economy is in. The north-west and south-east regions are not attainable, so they can be ignored. The economy is therefore attains equilibrium at the point of intersection by moving along the saddle path to that point. At this point there is no need for further changes in consumption and capital, and the economy is in equilibrium.

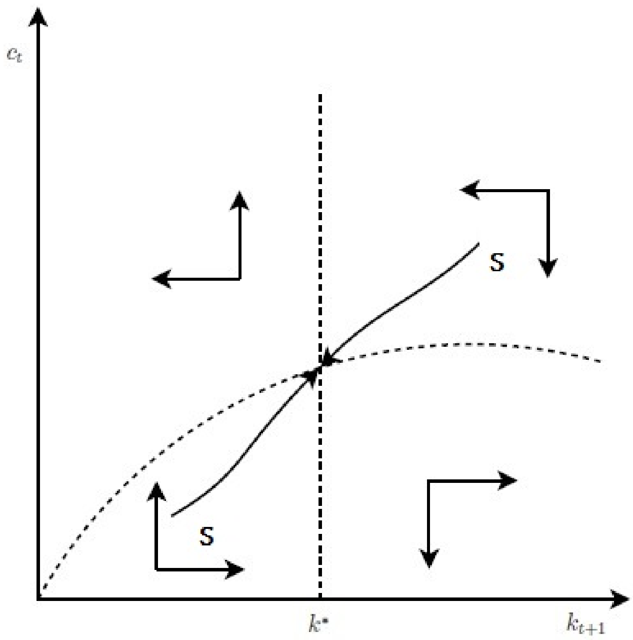

Figure 3: Combined diagram

Now assume that there is a change in the discount factor β. The new discount factor is $\tilde{\beta} > \beta$. From the expression for k^* we see that k^* must increase Hence, the zero motion line for consumption shifts to the right.

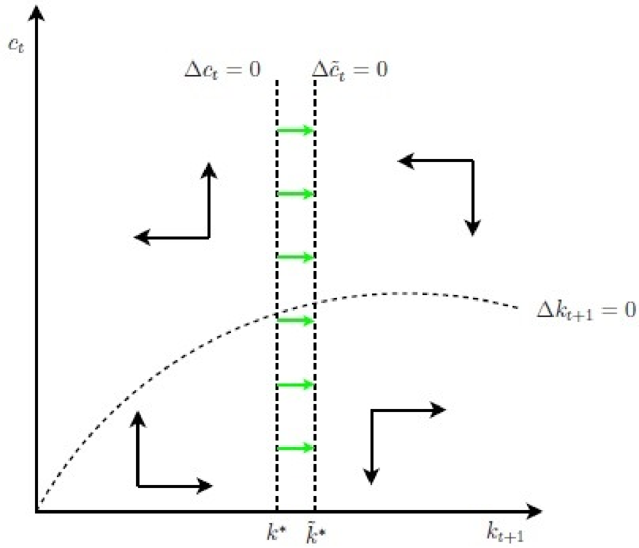

Figure 4: Change in β