Dynamic Macroeconomics

Problem Set 3

Non-linear difference equation systems like

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}(1 + \alpha A k_{t+1}^{\alpha-1} - \delta) \tag{1}$$
$$c_t + k_{t+1} = A k_t^{\alpha} + (1-\delta)k_t \tag{2}$$

can, in general, not be solved analytically. However, one can obtain an approximate solution to the system by linearizing it around the steady-state solution k^* and c^*. If one expresses the variables of the system in logarithms before doing the linearization, one calls this process log-linearization. This approach has the added benefit that the resulting expressions are easily interpreted as percentage deviations from the steady-state values. The intellectual background of the technique of log-linearization consists of some properties of the logarithmic and exponential function and one the property that any differentiable function can be approximated by a linear function. First, note that for any variable x_t the following is true:

$$x_t = x^* \frac{x_t}{x^*} = x^* e^{\ln\left(\frac{x_t}{x^*}\right)} = x^* e^{\hat{x}_t} \tag{3}$$

where $\hat{x_t} \equiv \ln\left(\frac{x_t}{x^*}\right)$ gives the percentage deviation of x_t from x^*. Second, note that for the exponential function the following approximation holds

$$e^x \approx 1 + x \tag{4}$$

which can be derived by taking a first-order Taylor expansion of e^x around the point $x^* = 0$:

$$e^x \approx e^{x^*} + e^{x^*}(x - x^*) = e^0 + e^0(x - 0) = 1 + x$$

Now we can apply these techniques to the equations of our equation system. For illustration we look only at condition (2). First use the transformation in (3) to get

$$\begin{aligned} c_t + k_{t+1} &= A k_t^{\alpha} + (1-\delta)k_t \\ c^* e^{\hat{c}_t} + k^* e^{\hat{k}_{t+1}} &= A\left[k^* e^{\hat{k}_t}\right]^{\alpha} + (1-\delta)k^* e^{\hat{k}_t} \end{aligned}$$

which can be approximated by (using the result in (4))

$$\begin{aligned} c^*[1+\hat{c}_t] + k^*[1+\hat{k}_{t+1}] &= A(k^*)^{\alpha}[1+\alpha\hat{k}_t] + (1-\delta)k^*[1+\hat{k}_t] \\ c^* + k^* + c^*\hat{c}_t + k^*\hat{k}_{t+1} &= A(k^*)^{\alpha} + (1-\delta)k^* + A(k^*)^{\alpha}\alpha\hat{k}_t + (1-\delta)k^*\hat{k}_t \\ c^*\hat{c}_t + k^*\hat{k}_{t+1} &= A(k^*)^{\alpha}\alpha\hat{k}_t + (1-\delta)k^*\hat{k}_t \\ \frac{c^*}{k^*}\hat{c}_t + \hat{k}_{t+1} &= A(k^*)^{\alpha-1}\alpha\hat{k}_t + (1-\delta)\hat{k}_t \\ \frac{c^*}{k^*}\hat{c}_t + \hat{k}_{t+1} &= [1 + \alpha A(k^*)^{\alpha-1} - \delta]\hat{k}_t \\ \frac{c^*}{k^*}\hat{c}_t + \hat{k}_{t+1} &= \frac{1}{\beta}\hat{k}_t \end{aligned}$$

where the last line uses the fact, which can be seen from (1), that in the steady-state the relationship $\frac{1}{\beta} = [1 + \alpha A(k^*)^{\alpha-1} - \delta]$ holds.